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REFINED FORMULAS FOR DETERMINATION OF THE INVERSE LAPLACE TRANSFORM USING FOURIER SERIES AND THEIR USE IN PROBLEMS OF HEAT CONDUCTION

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Refined formulas for determination of the original function from its Laplace transform which is represented in terms of Fourier series are given. For the cases where the asymptotic values of the function are known at large values of the argument, the inversion formulas are written in the form of a rapidly converging series. The results obtained are applied to solution of three-dimensional problems of nonstationary heat conduction.

I. We consider the problem of determination of the function $f(t)$ on the basis of its known Laplace transform $F(s)$. Let d denote such a constant that the function $F(s)$ is analytical when $\mathrm{Re}(s) > d$. Then for $0 < t < l$ we have the exact formula [1]

$$f(t) = \frac{1}{l} \exp(ct/l) \sum_{n=-\infty}^{\infty} F(s_n) \exp(2\pi nit/l) - R , \tag{1}$$

where $s_n = (c + 2\pi ni)/l$; l and c are the constants which satisfy the conditions $l > 0$ and $\mathrm{Re}(c) > 0$;

$$R = \sum_{n=1}^{\infty} \exp(-nc) f(t + nl) . \tag{2}$$

On neglect of the quantity R, relation (1) coincides with the known formula of an approximate inverse Laplace transform which is based on the representation of the original by Fourier series [2].

Approximate account for the quantity R. The quantity R in formula (2) can approximately be determined in the cases where an asymptotic expression for the function $f(t)$ is known when $t \Rightarrow \infty$. We denote the asymptotic value of the function $f(t)$ by $f_\infty(t)$ when $t \Rightarrow \infty$ and select l such that $f(t) \cong f_\infty(t)$ when $t > l$. Then

$$R = A(t, l, c) + \varepsilon ,$$

where

$$A(t, l, c) = \sum_{n=1}^{\infty} \exp(-nc) f_\infty(t + nl) , \quad \varepsilon = \sum_{n=1}^{\infty} \exp(-nc) [f(t + nl) - f_\infty(t + nl)] .$$

Here A is the known function. The quantity ε can also be made as small as is wished due to the parameters c and l. Then, formula (1), where the quantity R is approximately replaced by A, holds for the Laplace transform.

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TABLE 1. Temperature Distribution in a Body with an Ellipsoidal Cavity Heated by an Internal Heat Source

θ	1	2	3
0.012	45.88932	0.06583092	1.136163
0.024	47.62981	0.1907055	1.865293
0.036	48.33032	0.3370603	2.259732
0.048	48.73237	0.4666032	2.514371
0.06	49.01385	0.5759680	2.696226
0.072	49.25160	0.6681260	2.834654
0.084	49.50851	0.7464820	2.944862
0.096	49.90291	0.8138161	3.035894
0.108	50.95706	0.872142	3.114854
0.12	85.58970	0.9226981	3.208557

The series for determination of the function A can be summed up for many cases of practical importance. In particular, when $f_\infty = \text{const}$ we obtain $A = f_\infty/(\exp(c) - 1)$. In the more general case where $f_\infty = a + bt$ with $a = \text{const}$ and $b = \text{const}$, we find $A \cong \frac{1}{\exp(c) - 1}\left(f_\infty + \frac{bl \exp(c)}{\exp(c) - 1}\right)$. For the case of vibrational functions at $f_\infty(t) = a \exp(i\omega t)$ we have $A \cong \frac{\exp(i\omega t)}{\exp(c - i\omega l) - 1}$.

Improvement of series convergence. The series in formula (1) converges slowly, as a rule. We consider the case where $f(0)$, $f'(0)$, and $\int_0^\infty f''(t) \exp(-st)\, dt$ exist and the first two quantities are known. Then, allowing for the asymptotic transform $F(s_n) \Rightarrow \frac{f(0)}{s_n} + \frac{f'(0)}{s_n^2} + \ldots$ for $n \Rightarrow 0$, we can write relation (1) in terms of the rapidly converging series

$$f(t) = \frac{1}{l} \exp(ct/l) \sum_{n=-\infty}^{\infty} \tilde{F}_n \exp(2\pi nit/l) + \frac{1}{1 - \exp(-c)}\left[f(0) + lf'(0)\left(\frac{t}{l} + \frac{1}{\exp(c) - 1}\right)\right] - R\,, \tag{3}$$

where $\tilde{F}_n = F_n - \left(\frac{f(0)}{s_n} + \frac{f'(0)}{s_n^2}\right)$, with $\tilde{F}_n = O(n^{-3})$ for $n \Rightarrow \infty$.

If $f(t)$ is a monotonically decreasing function and c is a real quantity, then in all the formulas presented the error can be decreased by replacement of the constant c by $c + \pi i$. Then the estimate $0 < \varepsilon_1 < \exp(-c)\,[f_\infty - f(t + l)]$ (for $c > 0$) holds for the error of the approximate formula.

Sometimes the accuracy of the inversion formula can be improved considerably using the Runge rule. We consider the case where the asymptotic value of $f_\infty = \text{const}$ is known and the function $f(t)$ is monotonically increasing (decreasing) for $t > 1$. We write formula (3) in the form

$$f(t) = S(t, c) - \varepsilon(t, c)\,. \tag{4}$$

We set $c_1 = c_0$ and $c_2 = c_0 + \pi i$, where c_0 is a certain real number (in the problems of heat conduction it was taken that $c_0 \cong 2$–4). Then, for determination of the original function the formula and the estimate of the upper and lower bounds

$$f(t) \approx 0.5\,[S(t, c_1) + S(t, c_2)]\,, \quad S(t, c_1) \le f(t) \le S(t, c_2)\,. \tag{5}$$

hold. According to the first formula of (5), the original function is found with an error of $\cong O(\exp(-2c))$.

II. The above-given refined inversion formulas are efficient only in the case where additional *a priori* data on the original function are known. In particular, these quantities can be found in the problems of heat conduction. By virtue of this, we apply the results obtained to solution of the boundary-value problems of nonstationary heat conduction, i.e., we use the method of integral Laplace transformation with subsequent numerical inversion which is based on the above-given formulas. To solve the resultant Helmholtz equations use is made of the method of boundary integral equations [3].

We consider the problem of heat conduction for an infinite body with an ellipsoidal cavity which is heated by a concentrated heat source located at the point (x_0, y_0, z_0) in more detail. It is assumed that the temperature decreases at infinity, heat exchange between the cavity medium and the body occurs according to the Newton law, and the temperature of the medium is zero at the initial instant of time.

Table 1 gives the distribution of the quantity aT/Q relative to $\theta = a\tau$ for the case where the ellipsoid semiaxes are $r_1 = 0.1$, $r_2 = 0.15$, $r_3 = 0.12$, $x_0 = 0$, $y_0 = 0$, $z_0 = 0.17$, $\alpha/\lambda = 1$, and $c = 2$. The results for the points $A(0, 0, r_3)$, $B(0, 0, -r_3)$, and $C(r_1, r_2, 0)$ are presented in columns 1, 2, and 3, respectively; ω is the frequency of vibrations.

NOTATION

Q, intensity of the heat source; a, thermal diffusivity; λ, thermal conductivity; τ, time; T, temperature; α, heat-transfer coefficient; θ, variable; ω, frequency of vibrations.

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